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INTERACTION OF STRESS WAVES WITH CURVILINEAR TUNNEL CRACKS OF LONGITUDINAL SHEAR IN A HALF-SPACE*

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A steady wave process is considered in the half-space with tunnel-like curvilinear cracks under conditions of antiplane deformation. The ensuing boundary value problems are reduced to singular integro-differential equations which are realized numerically. If the crack tip reaches the half-space boundary, the kernel of the integro-differential equation contains, besides a moving singularity of the Cauchy type, a fixed singularity, and this sigificantly affects the pattern of longitudinal shear stress wave fields. This case is the subject of detailed investigation below. Certain singularities of such wave processes are pointed out, and the results of calculations of the dynamic stress intensity coefficient are presented.

Stresses in bodies subjected to pulsed loading were considered in /1/. Various dynamic problems of the theory of elasticity in the case of media with a rectilinear slit appeared in /2,3/, and methods of calculating electromagnetic wave diffraction and scatter are presented in /4/.

1. Statement of the problem. Consider the half-space $x_2 \geqslant 0$ containing K curvilinear tunnel-like cracks L_j parallel to the x_3 axis (Fig.1 a). Let the half-space boundary $x_2 = 0$ be either free of forces (stress $\tau_{23} = 0$ at $x_2 = 0$) or constrained (displacement $w = 0$ at $x_2 = 0$), a load harmonic in time $Z_n^\pm = \operatorname{Re}(e^{-i\omega t} Z^\pm)$ be specified at the edges of L_j, and let a monochromatic wave of longitudinal shear

$$w_0 = \operatorname{Re}(e^{-i\omega t} W_0),\ \gamma_2 = \omega/c_2,\ c_2 = \sqrt{\mu/\rho} \qquad (1.1)$$
$$W_0 = \tau \exp[-i\gamma_2(x_1\cos\beta_* - x_2\sin\beta_*)],\ \tau = \mathrm{const}$$

where ω is the angular frequency, c_2 is the propagation velocity of the shear wave, μ and ρ are the shear modulus and density of the medium material w_0 is the elastic displacement along the x_3 axis, and $\beta_* \neq 0$ is the angle between the normal to the wave front and the x_1 axis, emanate from infinity.

We assume that the load (variation) amplitude $Z^+ = -Z^- = Z$ satisfies on $L = \bigcup L_j$ the Hölder condition, and that L_j are simple nonintersecting smooth Liapunov arcs with Hölder continuous curvature /5/.

The singular field of longitudinal shear stress generated by external forces is expressed in terms of displacements by formulas /6/

$$\tau_{13} = \mu\frac{\partial w}{\partial x_1},\quad \tau_{23} = \mu\frac{\partial w}{\partial x_2},\quad w = \operatorname{Re}[e^{-i\omega t}(W + W_0)] \qquad (1.2)$$
$$\nabla^2 W + \gamma_2^2 W = 0$$

The quantity W_0 is defined in (1.1), and W defines the perturbed wave field which satisfies one of the indicated above boundary conditions at the half-space boundary and, also, on L a boundary condition of the form

$$e^{i\psi}\left[\frac{\partial}{\partial\zeta}(W + W_0)\right]^\pm + e^{-i\psi}\left[\frac{\partial}{\partial\bar\zeta}(W + W_0)\right]^\pm = \pm\frac{Z^\pm}{\mu},\quad \zeta\in L \qquad (1.3)$$

where the upper sign relates to the left-hand crack edge (when moving from its tip a_j to tip b_j), and ψ is the angle between the positive normal to the left-hand edge at point ζ and the x_1 axis.

Generalizing /7/, we represent the unknown function W in the form

$$W(x_1, x_2) = \frac{1}{2}\int_L p(\zeta)\left[\frac{\partial}{\partial z}H_0^{(1)}(\gamma_2 r) - A\frac{\partial}{\partial\bar z}H_0^{(1)}(\gamma_2 r_1)\right]d\zeta - \qquad (1.4)$$
$$\frac{1}{2}\int_L p(\zeta)\left[\frac{\partial}{\partial\bar z}H_0^{(1)}(\gamma_2 r) - A\frac{\partial}{\partial z}H_0^{(1)}(\gamma_2 r_1)\right]d\bar\zeta - AW_1$$

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$$W_1=\tau\exp\left[-i\gamma_2(x_1\cos\beta_*-x_2\sin\beta_*)\right],\ r=|z-\zeta|,\ r_1=|z-\bar{\zeta}|$$
$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x_1}-i\frac{\partial}{\partial x_2}\right),\quad \frac{\partial}{\partial\bar z}=\frac{1}{2}\left(\frac{\partial}{\partial x_1}+i\frac{\partial}{\partial x_2}\right),\ \zeta=\xi_1+i\xi_2$$

where $p(\zeta)=\{p_j(\zeta),\ \zeta\in L_j\}$ are the unknown functions, $H_\nu^{(1)}(x)$ is a Hankel function of order ν, $A=1$ in the case of constrained half-space and $A=-1$ for a half-space with boundary $x_2=0$ free of forces, and $\bar\zeta$ is a quantity complex conjugate of ζ. Using the readily derived formulas

$$\frac{\partial^n}{\partial z^n}H_0^{(1)}(\gamma r)=\left(-\frac{\gamma}{2}\right)^n e^{-in\alpha}H_n^{(1)}(\gamma r),\qquad r=|z-z_0| \tag{1.5}$$
$$\frac{\partial^n}{\partial\bar z^n}H_0^{(1)}(\gamma r)=\left(-\frac{\gamma}{2}\right)^n e^{in\alpha}H_n^{(1)}(\gamma r),\qquad \alpha=\arg(z-z_0)$$
$$\frac{\partial^2}{\partial z\,\partial\bar z}H_0^{(1)}(\gamma r)=-\frac{\gamma^2}{4}H_0^{(1)}(\gamma r),\ z=x_1+ix_2,\ z_0=x_{10}+ix_{20}$$

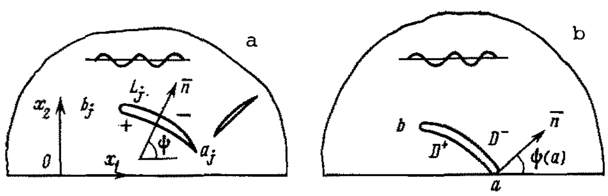

Fig.1

it is possible to show that relations (1.4) satisfy the Helmholtz equation (1.2) with related conditions at infinity $(x_2=+\infty)$, when $A=1$ giving automatically the conditions of constraints $w(x_1,0,t)=0$, and when $A=-1$ the condition $\tau_{23}(x_1,0,t)=0$.

2. Integral equations of the boundary value problem (1.3) in the case when cracks do not reach the half-space boundary. Differentiating function W of (1.4) using formulas (1.5), we obtain

$$\frac{\partial W}{\partial z}=\frac{1}{2\pi i}\int_L\frac{dp(\zeta)}{\zeta-z}+\int_L p(\zeta)K_1(\zeta,z)\,ds-A\frac{\partial W_1}{\partial z} \tag{2.1}$$
$$\frac{\partial W}{\partial\bar z}=-\frac{1}{2\pi i}\int_L\frac{dp(\zeta)}{\bar\zeta-\bar z}-\int_L p(\zeta)K_2(\zeta,z)\,ds-A\frac{\partial W_1}{\partial\bar z}$$
$$K_1(\zeta,z)=\frac{i\gamma_2^2}{8}e^{i\psi}\left[AH_0^{(1)}(\gamma_2r_1)+H_2(\gamma_2r)e^{-2i\alpha}\right]-$$
$$\frac{i\gamma_2^2}{8}e^{-i\psi}\left[H_0^{(1)}(\gamma_2r)+AH_2^{(1)}(\gamma_2r_1)e^{-2i\alpha_1}\right]$$
$$K_2(\zeta,z)=\frac{i\gamma_2^2}{8}e^{i\psi}\left[H_0^{(1)}(\gamma_2r)+AH_2^{(1)}(\gamma_2r_1)e^{2i\alpha_1}\right]-$$
$$\frac{i\gamma_2^2}{8}e^{-i\psi}\left[H_2(\gamma_2r)e^{2i\alpha}+AH_0^{(1)}(\gamma_2r_1)\right]$$
$$H_2(\gamma_2r)=\frac{4i}{\pi\gamma_2^2r^2}+H_2^{(1)}(\gamma_2r),\quad \alpha=\arg(z-\zeta),\quad \alpha_1=\arg(z-\bar\zeta)$$

where ds is an element of arc of contour of L.

Integration by parts was used in the derivation of formulas (2.1), which in this case is justified.

Indeed the displacement jump on L is

$$\Delta W=W^+-W^-=2p_j(\zeta),\qquad \zeta\in L_j\ (j=1,\ldots,k) \tag{2.2}$$

from which under the condition that cracks do not reach the half-space boundary we have

$$p_j(a_j)=p_j(b_j)=0\qquad (j=1,\ldots,k) \tag{2.3}$$

where a_j is the beginning and b_j the end of a crack L_j.

Passing in formulas (2.1) to limit values as $z\to\zeta_0\in L$ and substituting them in boundary conditions (1.3), after some transformations we obtain

$$\int_L p'(\zeta)g(\zeta,\zeta_0)\,ds+\int_L p(\zeta)G(\zeta,\zeta_0)\,ds=N(\zeta_0) \tag{2.4}$$
$$G(\zeta,\zeta_0)=\frac{\pi i\gamma_2^2}{4}\left[AH_0^{(1)}(\gamma_2r_{10})\cos(\psi+\psi_0)-H_0^{(1)}(\gamma_2r_0)\cos(\psi-\psi_0)+\right.$$
$$\left.H_2(\gamma_2r_0)\cos(\psi+\psi_0-2\alpha_0)-AH_2^{(1)}(\gamma_2r_{10})\cos(\psi-\psi_0+2\alpha_{10})\right]$$
$$p'(\zeta)=\frac{dp(\zeta)}{ds},\qquad r_0=|\zeta_0-\zeta|,\qquad r_{10}=|\zeta_0-\bar\zeta|$$
$$\alpha_0=\arg(\zeta_0-\zeta),\ \alpha_{10}=\arg(\zeta_0-\bar\zeta),\ \psi_0=\psi(\zeta_0),\ \zeta,\ \zeta_0\in L$$

where

$$g(\zeta, \zeta_0) = \operatorname{Im} \frac{e^{i\psi_0}}{\zeta - \zeta_0} \tag{2.5}$$

and in the case of a half-space constrained along its boundary $x_2 = 0$ and of a free half-space we, respectively, have

$$A = 1,\ N(\zeta_0) = \pi \left[\frac{z}{\mu} + \tau\gamma_2 \exp(-i\gamma_2 x_{10} \cos\beta_*) (e^{i\psi_0} \sin B^+ + e^{-i\psi_0} \sin B^-) \right] \tag{2.6}$$

$$A = -1,\ N(\zeta_0) = \pi \left[\frac{z}{\mu} + \tau i\gamma_2 \exp(-i\gamma_2 x_{10} \cos\beta_*) \right. \tag{2.7}$$
$$\left. (e^{i\psi_0} \cos B^+ + e^{-i\psi_0} \cos B^-) \right],\quad B^\pm = \gamma_2 x_{20} \sin\beta_* \pm \beta_*$$

In (2.4) the kernel $G(\zeta, \zeta_0)$ has a logarithmic singularity, and kernel $g(\zeta, \zeta_0)$ consists of a singular term (the Cauchy kernel) and of a term which cannot have more than a weak singularity /7/. Consequently, (2.4) represents a singular integro-differential equation with respect to function $p'(\zeta)$. It must be supplemented by conditions of the form

$$\int_{L_j} p'(\zeta)\, ds = 0 \qquad (j = 1, \ldots, k) \tag{2.8}$$

The system of Eqs.(2.4) and (2.8) enables us to fully determine the solution of the stated boundary value problem.

3. The half-plane with a crack reaching the boundary $x_2 = 0$. Let the beginning of the crack (point a) lie on the boundary of the half-space (Fig.1, b). In that case formula (1.4) remains valid. When passing to limit in formulas (2.1) a nonintegrable singularity appears at point $z = \zeta = a$. Hence we represent them

$$\frac{\partial W}{\partial z} = \frac{1}{2\pi i} \int_L \frac{dp(\zeta)}{\zeta - z} + \frac{A}{2\pi i} \int_L \frac{dp(\zeta)}{\bar\zeta - z} + \int_L p(\zeta) K_3(\zeta, z)\, ds + \tag{3.1}$$
$$\frac{p(a)}{2\pi i} \left(\frac{1}{a - z} + \frac{A}{\bar a - z} \right) - A \frac{\partial W_1}{\partial z},$$
$$\frac{\partial W}{\partial \bar z} = -\frac{1}{2\pi i} \int_L \frac{dp(\zeta)}{\bar\zeta - \bar z} - \frac{A}{2\pi i} \int_L \frac{dp(\zeta)}{\zeta - \bar z} - \int_L p(\zeta) K_4(\zeta, z)\, ds -$$
$$\frac{p(a)}{2\pi i} \left(\frac{1}{\bar a - \bar z} + \frac{A}{a - \bar z} \right) - A \frac{\partial W_1}{\partial \bar z}$$

Kernels $K_3(\zeta, z)$ and $K_4(\zeta, z)$ are obtained from $K_1(\zeta, z)$ and $K_2(\zeta, z)$, respectively, by substituting in the two last function $H_2(\gamma_2 r_1)$ for $H_2^{(1)}(\gamma_2 r_1)$.

Substituting limit values of derivatives in (3.1) as $z \to \zeta_0 \in L$ into the boundary condition (1.3) we obtain, after transformations the singular integro-differential equation in $p(\zeta)$ of the type (2.4), where

$$g(\zeta, \zeta_0) = \operatorname{Im} \left[\frac{e^{i\psi_0}}{\zeta - \zeta_0} + A \frac{e^{i\psi_0}}{\bar\zeta - \zeta_0} \right] \tag{3.2}$$

where $H_2(\gamma_2 r_{10})$ appears instead of $H_2^{(1)}(\gamma_2 r_{10})$.

For the constrained and free half-spaces we have, as previously, formulas (2.6) and (2.7), respectively. It was taken into account in the derivation of this equation that in the case of constrained half-space $p(a) = 0$.

Formula (3.2) shows that the first term in kernel $g(\zeta, \zeta_0)$ has a moving singularity of the Cauchy type, while the second term has a fixed singularity at point $\zeta = \zeta_0 = a$. The kernel $G(\zeta, \zeta_0)$ has a logarithmic singularity.

To find the order of the density singularity $p'(\zeta)$ at the crack tip a, we set

$$p'(\zeta) = \frac{dp}{ds} = \frac{\varphi(\zeta)}{(\zeta - a)^\sigma}, \quad \operatorname{Im}\sigma = 0, \quad 0 \leqslant \sigma < 1 \tag{3.3}$$

where function $\varphi(\zeta) \in H$ on arc $[a, b)$.

Using the well-known formulas for asymptotic values of integrals of the Cauchy type on the integration line /5/, we obtain

$$\int_L p'(\zeta) \operatorname{Im} \frac{e^{i\psi_0}}{\zeta - \zeta_0}\, ds = \varphi_1(\zeta_0) - \frac{\pi\varphi(a)}{(\zeta_0 - a)^\sigma} \operatorname{ctg}\pi\sigma \tag{3.4}$$
$$\int_L p'(\zeta) \operatorname{Im} \frac{e^{i\psi_0}}{\bar\zeta - \zeta_0}\, ds = \varphi_2(\zeta_0) + \frac{\pi\varphi(a)}{(\zeta_0 - a)^\sigma} \frac{\cos[2(\sigma - 1)\psi(a)]}{\sin\pi\sigma}$$

where $\psi(a)$ is the value of $\psi(\zeta)$ at point $\zeta = a$, and functions $\varphi_i(\zeta)$ can have at point $\zeta = a$ a singularity that is weaker than $(\zeta - a)^{-\sigma}$. Substituting Eq.(3.4) into Eqs.(2.4) and (3.2), multiplying the left- and right-hand sides of the obtained equality by $(\zeta_0 - a)^\sigma$, and passing then to limit as $\zeta_0 \to a$, we obtain the equation

$$A \cos [2(\sigma - 1)\psi(a)] - \cos \pi\sigma = 0 \tag{3.5}$$

The analysis of solution of this equation in the case of $A = 1$ yields

$$\sigma = \psi(a)[\psi(a) + \pi/2]^{-1}, \ 0 \leqslant \psi(a) < \pi/2, \quad \sigma = \psi(a)[\psi(a) - \pi/2]^{-1}, \ -\pi/2 < \psi(a) \leqslant 0 \tag{3.6}$$

There are no solutions of Eq. (3.5) in the case of $A = -1$ which satisfy the condition $0 < \sigma < 1$.

Thus, if the half-space is constrained along the boundary $x_2 = 0$ and the crack tip a reaches the boundary, the density $p'(\zeta)$ has at point a a power singularity of order σ, which is determined by formulas (3.3) and (3.6). Obviously $0 \leqslant \sigma < {}^1/_2$. If, however, the half-plane boundary is free of forces, function $p'(\zeta)$ is bounded at the tip a.

4. Asymptotic values of stresses at the crack tip.

If the crack does not reach the boundary $x_2 = 0$, the density $p'(\zeta)$ has at the tip a singularity of the square root type /8/. Let us set

$$\zeta = \zeta(\beta), \ \zeta_0 = \zeta(\beta_0), \ -1 \leqslant \beta, \beta_0 \leqslant 1, \quad p'(\zeta) = \frac{\Omega_0(\beta)}{s'(\beta)\sqrt{1-\beta^2}}, \quad s'(\beta) = \frac{ds}{d\beta} > 0, \quad \Omega_0(\beta) \in H[-1, 1] \tag{4.1}$$

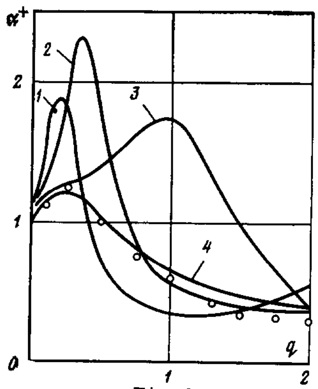

Fig.2

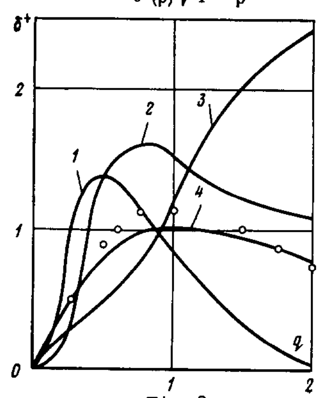

Fig.3

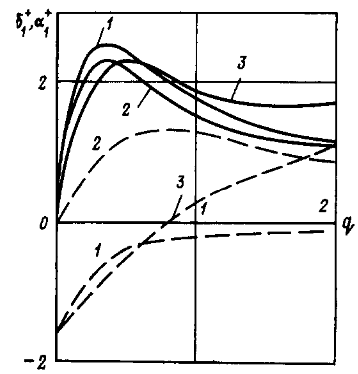

Fig.4

Taking advantage of the behavior of the Cauchy type integrals, appearing in (2.1) in the neighborhood of the ends of the integration line L /5/ and using formulas (4.1) and (1.2), we obtain the following asymptotic values of stresses in the crack tip neighborhood:

$$\tau_{13} - i\tau_{23} = \mp \frac{\mu}{\sqrt{2rs'(\pm 1)}} \exp\left[\frac{i}{4}(\pm\pi - 2\psi(\pm 1) - 2\Theta)\right] \times \tag{4.2}$$

$$\operatorname{Re}[e^{-i\omega t}\Omega_0(\pm 1)], \quad s'(\pm 1) = \left.\frac{ds}{d\beta}\right|_{\beta = \pm 1}$$

$$r = |z - c|, \quad \Theta = \arg(z - c)$$

where the upper sign relates to the crack tip $c = b$ and the lower to the tip $c = a$.

Along the continuation beyond the crack tip we have

$$\tau_{13} - i\tau_{23} = \frac{\mu e^{-i\psi(\pm 1)}}{\sqrt{2rs'(\pm 1)}} \operatorname{Re}[e^{-i\omega t}\Omega_0(\pm 1)] \tag{4.3}$$

$$\tau_n = \tau_{13}\cos\psi(\pm 1) + \tau_{23}\sin\psi(\pm 1) = \frac{\mu}{\sqrt{2rs'(\pm 1)}} \operatorname{Re}[e^{-i\omega t}\Omega_0(\pm 1)]$$

Thus the highest shear stress occurs on the small area of crack extension beyond its tip. The dynamic stress intensity coefficient is determined by the formula /3/

$$k_3 = \sqrt{2\pi r}\,\tau_n = \mu \sqrt{\frac{\pi}{s'(\pm 1)}} \operatorname{Re}[e^{-i\omega t}\Omega_0(\pm 1)] \tag{4.4}$$

Let us now consider the case when the crack tip a reaches the half-space $x_2 = 0$. If the boundary is constrained, then, according to the method of singular solutions /3/ in the sector of acute angle, stresses τ_{13}, τ_{23} are finite, while in the complementary sector they have a power singularity whose order σ is defined in (3.6). When the crack reaches the constrained boundary at a straight angle, or the boundary is free of forces, stresses at point a are finite.

5. Results of calculations.

We considered a parabolic crack whose parametric equations are

$$\xi_1 = p_1\beta, \ \xi_2 = p_2 + p\beta^2, \ -1 \leqslant \beta \leqslant 1 \tag{5.1}$$

The integral equation (2.4) was reduced, with (4.1) and (5.1) taken into account, to a system of linear algebraic equations in terms of function $\Omega_0(\beta)$ at the interpolation nodes in conformity with the procedures in /9/. Results of calculations are shown in Figs.2–4.

The dependence of $\alpha^+ = \mu|\Omega_0(1)|/(Z\sqrt{ls'(1)})$ (Fig.2) and of $\delta^+ = \arg\Omega_0(1)$ (Fig.3) on the variable $q = \gamma_2^2 l^2/4$ (l is the crack half-length) for the case $\tau = 0$, $Z \neq 0$, $p_1 = 1$. Curve *1* corresponds to $p_2 = 1$, $A = -1$, $p = 0.05$; curve *2* to $p_2 = 1$, $A = -1$, $p = 1$; and curve *3* to $p_2 = 1$, $A = 1$, $p = 1$. Curve *4* relates to a "straight" crack in an unbounded medium ($p = 0.05$, $A = 0$). For comparison data from /10/ are shown there by small circles.

When the quantities α^+, δ^+ are known it is possible to calculate the dynamic intensity coefficient k_3 using formula

$$k_3 = Z\sqrt{\pi l}\alpha^+ \cos(\omega t - \delta^+)$$

The dependence of $\alpha_1^+ = |\Omega_0(1)|\sqrt{l}/(\tau\sqrt{s'(1)})$ (solid lines) and $\delta_1^+ = \arg\Omega_0(1)$ (dash lines) on q are shown in Fig.4 for the case of $Z = 0$, $\tau \neq 0$, $p_1 = 1$, , with $p_2 = \pi/\gamma_2$, $p = 10^{-4}$, $A = 1$ (curves *1*): $p_2 = \pi/(2\gamma_2)$, $p = 10^{-4}$, $A = -1$ (curves *2*), and $p_2 = \pi/\gamma_2$, $p = 1$, $A = 1$ (curves *3*). The dynamic intensity coefficient is in this case calculated using formula

$$k_3 = \mu\tau\sqrt{\pi/l}\alpha_1^+ \cos(\omega t - \delta_1^+)$$

The above data show the substanial effect of crack length on coefficient k_3.

The actuality of dynamic problems of the theory of elasticity of the half-space was pointed out by A.A. Il'iushin at the All-Union Conference on the Theory of Elasticity (Erevan', 1979) in connection with detection of defects.

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